The flatness properties of diagonal S-posets

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Abstract This paper investigates the flatness properties of diagonal S-posets $D(S)$ over pomonoids. We characterize pomonoids for which $D(S)$ satisfies various flatness conditions, including (E^*), (P_E), (P'),and (E'). Additionally, we explore the transfer of these properties from products of S-posets to their components and provide conditions under which these properties transfer from S-posets to their products.

Key Words: Pomonoid S-posets; Flatness properties; Diagonal S-posets; Product of pomonoids

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1. Introduction and preliminaries

A left S-act over a monoid S is a nonempty set B equipped with a mapping $S \times B \longrightarrow B$, denoted $(s, b) \mapsto sb$, satisfying $s(tb) = (st)b$ and $1 \cdot b = b$ for all $s, t \in S$ and $b \in B$. Right S-acts are defined similarly. The class of all left and right S-acts is denoted by $\mathbf{S} - \mathbf{Act}$ and $\mathbf{Act} - \mathbf{S}$, respectively. A subset C of a left S-act B is an S-subact if it is closed under the action of S. Any left ideal of S is a subact of ${}_SS$, and similarly, any right ideal is a subact of S_S.

The diagonal S-act $D(S)$ is defined as the right S-act $S \times S$ with componentwise S-action. Diagonal S-acts form a special class of S-acts, and their flatness properties have been extensively studied in [1, 5, 13, 15]. These studies aim to determine conditions under which the diagonal S-act possesses certain flatness properties and to explore the separation of these properties.

A pomonoid is a monoid S equipped with a partial order compatible with the binary operation. A left S-poset is a partially ordered set A with an S-action that is monotonic in each variable, satisfying $s(ta) = (st)a$ and $1 \cdot a = a$ for all $s, t \in S$ and $a \in A$.

An S-subposet B of an S-poset A is strongly convex if $a \leqslant b$ and $b \in B$ imply $a \in B$. An S-poset A is decomposable if it can be expressed as a disjoint union of nonempty

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strongly convex S-subposets A_1 and A_2; otherwise, it is indecomposable.

We know that x, y of an ordered set X are comparable if $x \leqslant y$ or $y \leqslant x$ and denote this relation by $x \nparallel y$.

Lemma 1.1 ([18] Proposition 2.6) *Let A be an S-poset and $a, b \in A$. Then a and b belong to the same strongly convex indecomposable component of A if and only if there exist $a_1, \cdots, a_n \in A$, $s_1, t_1, \cdots, s_n, t_n \in S$ such that*

$$a \nparallel s_1 a_1,$$

$$t_1 a_1 \nparallel s_2 a_2,$$

$$t_2 a_2 \nparallel s_3 a_3,$$

$$\vdots$$

$$t_{n-1} a_{n-1} \nparallel s_n a_n,$$

$$t_n a_n \nparallel b.$$

An S-morphism $\phi : A \longrightarrow B$ between left S-posets is called an S-poset morphism if it is order-preserving.

Flatness properties of S-posets were first studied by Fakhruddin in the 1980s [6,7], and further developed in [2,3,4,19]. In [2,16], properties such as (po-)flatness, (po-)torsion freeness, Condition (P), and (P_w)were introduced. An element c of a pomonoid S is right po-cancellable if $sc \leqslant sc'$ implies $s \leqslant s'$ for all $s, s' \in S$. An S-poset A_S is po-torsion free if $ac \leqslant a'c$ implies $a \leqslant a'$ for all $a, a' \in A$ and right po-cancellable $c \in S$.

An S-poset A_S is po-flat if for every embeddings ${}_SB \to_S C$ of left S-posets, the induced map $A \otimes_S B \to A \otimes_S C$ is order-injective. A_S is (principally) weakly po-flat if the functor $A_S \otimes -$ preserves order embeddings of (principal) left ideals of pomonoid S into S.

Condition (P) is satisfied by an S-poset if, for all $a, a' \in A_S$ and $s, s' \in S$, $as \leqslant a's'$ implies the existence of $a'' \in A_S$, $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us \leqslant vs'$. Similarly,

Condition (P_w) requires that $as \leqslant a's'$ implies the existence of $a'' \in A_S$, $u, v \in S$ such that $a \leqslant a''u$, $a''v \leqslant a'$ and $us \leqslant vs'$.

The diagonal S-poset $S \times S$ is denoted by $D(S)$. In [12], Roghaieh characterized pomonoids over which direct products of (po-)torsion free, principally weakly, and weakly (po-)flat S-posets retain these properties. This paper extends the study of flatness properties of diagonal S-posets.

In this paper, we continue discussing flatness properties of the diagonal S-posets. Some results on S-acts can be also obtained as applications of the results in this paper.

2. (principally) weakly po-flat, Condition (E^*), Condition (P') and Condition (E') of diagonal S-posets $D(S)$

This section investigates the (principally) weakly po-flatness of diagonal S-posets $D(S)$. We characterize pomonoids for which $D(S)$ satisfies Conditions (E^*). Additionally, we address the transfer of properties such as (P_E), (PWP_E), (P') and (E') from S-posets to their (finite) products.

Definition 2.1 *A pomonoid S is finite (principally) weakly po-coherent if finite products of (principally) weakly flat S-posets are (principally) weakly flat.*

It is mentioned in [16 Theorem 3.12], a weakly po-flat left S-poset A is equality to it is principally weakly po-flat and satisfies the following Condition (W) which means if $xa \leqslant ya'$ for $a, a' \in A$, $x, y \in S$, then there exist $a'' \in A$, $p \in xS$, $q \in yS$ such that $p \leqslant q$, $xa \leqslant pa''$, $qa'' \leqslant ya'$.

Next, let's memorize some notations. For any $(p, q) \in D(S)$, $\widehat{S(p,q)} = \{(u, v) \in D(S) | \exists w \in S, u \leqslant wp, wq \leqslant v\}$. If $Ss \cap (St] \neq \varnothing$, write $H(s, t) = \{(as, a't) | as \leqslant a't\}$. We know the diagonal S-poset $D(S)$ is weakly po-flat if and only if it is principally weakly po-flat and $Ss \cap (St] \neq \varnothing$ or for each $(as, a't)$ and $(bs, b't)$ in $H(s, t)$, there exists $(p, q) \in H(s, t)$ such that $(as, a't), (bs, b't) \in \widehat{S(p,q)}$.

Theorem 2.2 *A pomonoid S is finite weakly po-coherent if and only if $D(S)$ is weakly po-flat.*

Proof We suppose that the diagonal S-poset $D(S)$ is weakly po-flat and consider two weakly po-flat S-posets A_S and B_S. $(A_S \times B_S)$ is also principally weakly po-flat since $D(S)$ is principally weakly po-flat. Thus we claim that $(A_S \times B_S)$ satisfies Condition (W). Let $(a, b)s \leqslant (a', b')t$ for $a, a' \in A_S$, $b, b' \in B_S$ and $s, t \in S$. From weak po-flatness of A_S and B_S, there exist $a'' \in A_S$, $b'' \in B_S$, $p_1, p_2 \in Ss$, $q_1, q_2 \in St$ such that $p_1 \leqslant q_1$, $p_2 \leqslant q_2$, $as \leqslant a''p_1$, $a''q_1 \leqslant a't$ and $bs \leqslant b''p_2$, $b''q_2 \leqslant b't$. By statement above, there exists $(p, q) \in H(s, t)$ such that $(as, a't), (bs, b't) \in \widehat{S(p,q)}$. That is, $as \leqslant wp$, $wq \leqslant a't$ and $bs \leqslant w'p$, $w'q \leqslant b't$ for some $w, w' \in S$, so $(a, b)s \leqslant (w, w')p$, $(w, w')q \leqslant (a', b')t$ as required. □

In [16], the author introduced the concept of Condition (PF) on S-posets, and proved that A_S is strongly flat if and only if A_S satisfies Condition (PF), as follows Lemma 2.3

Lemma 2.3 ([16] Proposition 2.3) *Let B_S is a left S-poset. B_S is strongly flat if and only if it is satisfies Condition (PF) which $sb \leqslant s'b'$, $tb \leqslant t'b'$ for $b, b' \in B_S$, $s, s', t, t' \in S$, then there exist $b'' \in B_S$, $u, v \in S$ such that $b = ub''$, $b' = vb''$, $su \leqslant s'v$ and $tu \leqslant t'v$.*

In fact, the reference to this conclusion in [20] is not true, Condition (PF) is not equivalent to strong flatness and it is just a generalization of strong flatness. Furthermore, the author introduced a new concept from [20].

Condition (E^*) is satisfied that if $as \leqslant at$, then there exist $a' \in A_S$, $u, v \in S$ such that $a = a'u = a'v$, $us \leqslant vt$ and $u \leqslant v$. Here we give a characterization of pomonoids for the diagonal S-posets $D(S)$ satisfying Condition (E^*).

Theorem 2.4 *For any pomonoid S, the following conditions are equivalent:*

(1) If $A_1, \cdots, A_n$ satisfy Condition (E^*), then $A_1 \times \cdots \times A_n$ satisfies Condition (E^*);

(2) The diagonal S-poset $D(S)$ satisfies Condition (E^*);

(3) For any $a, b \in S$, $(a,a),(b,b) \in L(s,s')$, there exist $u, v \in S$ with $(u,v) \in L(s,s') \cap L(1,1)$ such that $(a,a),(b,b) \in S(u,v)$.

Proof (1) $\Rightarrow$ (2) Obviously.

(2) $\Rightarrow$ (3) Let the diagonal S-poset $D(S)$ satisfies Condition (E^*) and $(a,a),(b,b) \in L(s,s')$, $a, b \in S$. Then $as \leqslant as'$, $bs \leqslant bs'$, then implies $(a,b)s \leqslant (a,b)s'$. By Condition (E^*), there exist $(a',b') \in D(S)$ and $u, v \in S$ such that $(a,b) = (a',b')u = (a',b')v$, $us \leqslant vs'$ and $u \leqslant v$. Thus, $(u,v) \in L(s,s') \cap L(1,1)$, $(a,a),(b,b) \in S(u,v)$.

(3) $\Rightarrow$ (1) Assume that $A_1, \cdots, A_n$ satisfy Condition (E^*). Let $(a_1, \cdots, a_n)s \leqslant (a_1, \cdots, a_n)s' \in \Pi_{i\in I} A_i$, $a_i, a_i' \in A_i$, $i \in I$, $s, s' \in S$. Then $a_i s \leqslant a_i s'$ and by Condition (E^*), there exist $a_i' \in A_i$, $p_i, q_i \in S$ such that $a_i = a_i' p_i = a_i' q_i$, $p_i s \leqslant q_i s'$, It follow that $(p_i, q_i) \in L(s,s')$. By assumption, there exists $(p,q) \in D(S)$ with $(p,q) \in L(s,s') \cap L(1,1)$ such that $(p_i,q_i) \in S(p,q)$. That is, $(p_i,q_i) = w_i(p,q)$ for some $w_i \in S$, and $ps \leqslant qs'$, $p \leqslant q$. Then

$$\begin{aligned}(a_1, \cdots, a_n) &= (a_1', \cdots, a_n')p_i = (a_1', \cdots, a_n')q_i \\ &= (a_1' w_1, \cdots, a_n' w_n)p = (a_1' w_1, \cdots, a_n' w_n)q.\end{aligned}$$

as required. □

Proposition 2.5 *For any pomonid S, the followings are equivalent:*

(1) The direct product of every nonempty family of right S-posets satisfying Condition (E^*) satisfies Condition (E^*);

(2) $(S^\Gamma)_S$ satisfies Condition (E^*) for every non-empty set Γ;

(3)For every $s, s' \in S$, the set $L(s,s') \cap L(1,1)$ is either empty or a cyclic left S-poset.

Proof (1) $\Rightarrow$ (2) Obviously.

(2) $\Rightarrow$ (3) Suppose that $L(s,s') \neq \varnothing$, $s, s' \in S$. Let $\vec{u}$ be the element of S^Γ whose γth component is u_γ and $(u_\gamma, u_\gamma) \in L(s,s')$. Then $\vec{u}s \leqslant \vec{u}s'$ in S^Γ. By assumption, S^Γ satisfies Condition (E^*), there exist $\vec{z} \in S^\Gamma$ and $p, q \in S$ such that $\vec{u} = \vec{z}p = \vec{z}q$, $ps \leqslant qs'$ and $p \leqslant q$. Thus $(p,q) \in L(s,s') \cap L(1,1)$, so that $(p,q) = (u_j, v_j)$ for some $j \in \Gamma$. If $\gamma \in \Gamma$, then $(u_\gamma, u_\gamma) = z_\gamma(p,q) = z_\gamma(u_j, v_j)$ where z_γ is the γth component of $\vec{z}$. It follows that $L(s,s') \cap L(1,1)$ is a cyclic left S-poset.

(3) $\Rightarrow$ (1) Let $A = \Pi_{i\in I} A_i$ be a product of right S-posets satisfying Condition (E^*). Suppose that $\vec{x}s \leqslant \vec{x}s'$ where $s, s' \in S$ and $\vec{x} = (x_i) \in A$. For each $i \in I$, $x_i s \leqslant x_i s'$ and as A_i satisfies Condition (E^*) there exist $z_i \in A_i$ and $u_i, v_i \in S$ such that $x_i = z_i u_i = z_i v_i$, $u_i s \leqslant v_i s'$ and $u_i \leqslant v_i$. So $(u_i, v_i) \in L(s,s') \cap L(1,1)$ and by assumption, $(u_i, v_i) = r_i(p,q)$ for some $r_i \in S$, $(p,q) \in L(s,s') \cap L(1,1)$. We have $x_i = z_i r_i p = z_i r_i q$, $ps \leqslant qs'$ and $p \leqslant q$. Putting $\vec{w} = (z_i r_i)_{i\in I} \in A$, then $\vec{x} = \vec{w}p = \vec{w}q$ as required. □

Next, we demonstrate that Condition (E^*) transfers from S-posets to their products. Recall that a pomonoid T is called a left pogroup if, for all $a, b \in T$ there exists a unique $t \in T$ such that $ta = b$.

Proposition 2.6 *Let $S = T^1$, where T is a left pogroup. If A_i for each $i \in I$ satisfies Condition (E^*), then $\Pi_{i\in I}A_i$ also satisfies Condition (E^*).*

Proof Suppose $(a_i)_I s \leqslant (a_i)_I s'$ where $(a_i)_I \in \Pi_{i\in I}A_i$ and $s, s' \in S$. This implies that $a_i s = a_i s'$ for all $i \in I$. According to the given assumption, for each $i \in I$ there exist $a'_i \in A$, $u_i, v_i \in S$ such that $a_i = a'_i u_i = a'_i v_i$, $u_i \leqslant v_i$ and $u_i \leqslant v_i$. Now, consider two cases. If $u_j = v_j = 1$ for some $j \in I$, then $s \leqslant s'$. Otherwise, fix $k \in I$. Then for every $j \in I$, $j \neq k$, there exists $x_j \in T$ such that $u_j = x_j u_k$ and $v_j = y_j v_k$ for $y_j \in T$. Let $b_k = a'_k$ and $b_j = a' x_j = a' y_j$ for every $j \neq k$. Then, $(a_i)_I = (a' u_i)_I = (b_i)_I u_k$ and $(a_i)_I = (a' v_i)_I = (b_i)_I v_k$, which shows that $\Pi_{i\in I}A_i$ satisfies Condition (E^*). □

Now we address the question of when Condition (P_E) is transferred from products of S-posets to their individual components. We define that an S-poset A_S satisfies *Condition* (P_E), when for any $a, a' \in A_S$, $s, s' \in S$, the relation $as \leqslant a's'$ implies the existence $a'' \in A_S$, $u, v \in S$, $e, f \in E(S)$ such that $ae = a''ue$, $a'f = a''vf$, $es = s$, $fs' = s'$ and $us \leqslant vs'$.

Recall from [2] a pomonoid S called weakly right reversible in case $Ss \cap (St] \neq \varnothing$ for all $s, t \in S$, that is, for all $s, t \in S$, there exist $u, u' \in S$ such that $us \leqslant u't$. Then we can show the following Theorem 2.7.

Theorem 2.7 *Let S be a pomonoid. The following conditions are equivalent:*

(1) If $\Pi_{i\in I}A_i$ satisfies Condition (P_E), then each A_i satisfies also Condition (P_E);

(2) Θ_S satisfies Condition (P_E);

(3) S is a weakly right reversible pomonoid.

Proof (1) ⇒ (2) Since $S \times \Theta_S \cong S$, this implication is straightforward.

(2) ⇒ (3) The proof is similar to [8, Theorem 6.2].

(3) ⇒ (1) Assume that $\Pi_{i\in I}A_i$ satisfies Condition (P_E) and $a_i s \leqslant a'_i t$ where $a_i, a'_i \in A_S$, $s, t \in S$. Since S is weakly right reversible pomonoid, there exist $u_1, v_1 \in S$ such that $u_1 s \leqslant v_1 t$. Fix an element $a_j \in A_j$ for $j \neq i$, we define

$$c_j = \begin{cases} a_j u_1, & \text{if } j \neq i \\ a_i, & \text{if } j = i \end{cases} \quad \text{and} \quad d_j = \begin{cases} a_j v_1, & \text{if } j \neq i \\ a'_i, & \text{if } j = i \end{cases}.$$

Then $(c_j)_I s \leqslant (d_j)_I t$, according to the given assumption, there exist $(a''_j)_I \in \Pi_{i\in I}A_i$, $u, v \in S$, $e, f \in E(S)$ such that $(c_j)_I e = (a''_j)_I ue$, $(d_j)_I f = (a''_j)_I vf$, $es = s$, $ft = t$ and $us \leqslant vt$. Thus, $a_i e = a''_i ue$, $a'_i f = a''_i vf$. From these relations, it can be concluded that A_i satisfies Condition (P_E). □

Next, we will expound on certain pomonids with respect to Condition (P_E) and (PWP_E). As per the definition in [9] an S-poset is termed weakly locally cyclic when every finitely generated S-subposet of A is contained in a cyclic S-poset. A principal left ideal of S that also exhibits the property of being weakly locally cyclic is referred to as a weakly locally principal left ideal. According to the reference [12], the set $L(a, b) := \{(u, v) \in S \times S | ua \leqslant$

$vb\}$ is a left S-subposet of $D(S)$, and the set $l(a,b) := \{u \in S | ua \leqslant vb\}$ is a left ideal of S.

Definition 2.8 *The set $L(x,y)$ is said to be weakly locally cyclic idempotent if, for every finite subset $\{(x_1,y_1),\cdots,(x_n,y_n)\}$ of $L(x,y)$, $x,y \in S$, there exist $e_1,\cdots,e_{n-1},f_1,\cdots,f_{n-1} \in E(S)$, $u,v \in S$ such that $e_jx = x$, $f_jy = y$, $(u,v) \in L(x,y)$ and $(x_ie_1\cdots e_{n-1}, y_if_1\cdots f_{n-1}) \in S(ue_{n-1}, vf_{n-1})$, for $1 \leqslant j \leqslant n-1$, $1 \leqslant i \leqslant n$.*

Lemma 2.9 *The set $L(x,y)$ is weakly locally cyclic idempotent for $x,y \in S$ if and only if for every finite subset $\{(x_1,y_1),(x_2,y_2)\}$ of $L(x,y)$, there exist $e,f \in E(S)$, $u,v \in S$ such that $ex = x$, $fy = y$, $(u,v) \in L(x,y)$ and $(x_1e, y_1f),(x_2e,y_2f) \in S(ue,vf)$.*

Proof **Necessity.** The proof is clear.

Sufficiency. By using induction on n. Let $\{(x_1,y_1),\cdots,(x_n,y_n)\} \subseteq L(x,y)$. According to assumption, there exist $e_1,\cdots,e_{n-2},f_1,\cdots,f_{n-2} \in E(S)$, $w,z \in S$ such that $e_kx = x$, $f_ky = y$, $(w,z) \in L(x,y)$ and $(x_je_1\cdots e_{n-2}, y_jf_1\cdots f_{n-2}) \in S(we_{n-2}, zf_{n-2})$ for $1 \leqslant k \leqslant n-2$, $1 \leqslant j \leqslant n-1$. Thus, for each $1 \leqslant j \leqslant n-1$, we have that

$$(x_je_1\cdots e_{n-2}, y_jf_1\cdots f_{n-2}) = p_j(we_{n-2}, zf_{n-2})$$

for some $p_j \in S$. Since $(x_n,y_n) \in L(x,y)$, so

$$x_nx \leqslant y_ny \;\Rightarrow\; x_ne_1x \leqslant y_nf_1y \;\Rightarrow\; x_ne_1e_2x \leqslant y_nf_1f_2y$$

$$\Rightarrow\; \cdots \;\Rightarrow\; x_ne_1\cdots e_{n-2}x \leqslant y_nf_1\cdots f_{n-2}y,$$

it follows that $(x_ne_1\cdots e_{n-2}, y_nf_1\cdots f_{n-2}) \in L(x,y)$. Since $(w,z) \in L(x,y)$, which implies $(we_{n-2}, zf_{n-2}) \in L(x,y)$, by assumption, there exist $e_{n-1},f_{n-1} \in E(S)$, $u,v \in S$ such that $e_{n-1}x = x$, $f_{n-1}y = y$, $(u,v) \in L(x,y)$ and

$$(x_ne_1\cdots e_{n-2}e_{n-1}, y_nf_1\cdots f_{n-2}f_{n-1}),(we_{n-2}e_{n-1}, zf_{n-2}f_{n-1}) \in S(ue_{n-1}, vf_{n-1}).$$

And then

$$we_{n-2}e_{n-1} = que_{n-1},\;\; zf_{n-2}f_{n-1} = qvf_{n-1}$$

for some $q \in S$. And for any $1 \leqslant j \leqslant n-1$, $e_jx = x$, $f_jy = y$, $(u,v) \in L(x,y)$ and

$$(x_je_1\cdots e_{n-2}, y_jf_1\cdots f_{n-2}) = p_j(we_{n-2}, zf_{n-2}).$$

Thus, $x_je_1\cdots e_{n-2} = p_jwe_{n-2}$ implies

$$x_je_1\cdots e_{n-2}e_{n-1} = p_jwe_{n-2}e_{n-1} = p_jque_{n-1}$$

and from $y_jf_1\cdots f_{n-2} = p_jzf_{n-2}$, imples

$$y_jf_1\cdots f_{n-2}f_{n-1} = p_jzf_{n-2}f_{n-1} = p_jqvf_{n-1}.$$

So for any $1 \leqslant k \leqslant n$,

$$(y_ke_1\cdots e_{n-1}, y_kf_1\cdots f_{n-1}) \in S(ue_{n-1}, vf_{n-1}),$$

as required. □

Theorem 2.10 *Let S be a pomonoid. The following conditions are equivalent:*

(1) If $A_1, \cdots, A_n$ satisfy Condition (P_E) where $n \in \mathbb{N}$, then $\Pi_{i=1}^n A_i$ satisfies Condition (P_E);

(2) If A_S and B_S satisfy Condition (P_E), then $(A_S \times B_S)$ satisfies Condition (P_E);

(3) The diagonal S-poset $D(S)$ satisfies Condition (P_E);

(4) For any $a, b \in S$, the set $L(a, b)$ is either empty or else a weakly locally cyclic idempotent left S-poset.

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (4) Consider $a, b \in S$, let $(x, y), (z, w) \in L(a, b)$. This implies that $(x, z)a \leqslant (y, w)b$, according to relevant properties, there exist $e, f \in E(S)$, $u, v \in S$, $(a'', b'') \in D(S)$ such that

$$(x, z)e = (a'', b'')ue, \ (y, w)f = (a'', b'')vf, \ ea = a, \ fb = b, \ ua \leqslant vb.$$

Consequently, we have $(u, v) \in L(a, b)$ and $(xe, yf), (ze, wf) \in S(ue, vf)$, by Lemma 2.9, $L(a, b)$ a weakly locally cyclic idempotent left S-poset.

(4) $\Rightarrow$ (1) Assume that A_i satisfies Condition (P_E) for $1 \leqslant i \leqslant n$. Consider $(a_1, \cdots, a_n)x \leqslant (b_1, \cdots, b_n)y$ where $(a_1, \cdots, a_n), (b_1, \cdots, b_n) \in \Pi_{i=1}^n A_i$ and $x, y \in S$. This implies that $a_i x \leqslant b_i y$ and Since Condition (P_E), there exist $e_i, f_i \in E(S)$, $u_i, v_i \in S$, $d_i \in A_i$ such that

$$a_i e_i = d_i u_i e_i, \ b_i f_i = d_i v_i f_i, \ e_i x = x, \ f_i y = y, \ u_i x \leqslant v_i y.$$

Consequently, $(u_i, v_i) \in L(x, y)$. By assumption, there exist $g_1, \cdots, g_{n-1}, h_1, \cdots, h_{n-1} \in E(S)$, $p, q \in S$ such that $g_i x = x$, $h_i y = y$, $(p, q) \in L(x, y)$, $(u_j g_1 \cdots g_{n-1}, v_j h_1 \cdots h_{n-1}) \in S(pg_{n-1}, qh_{n-1})$ where $1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n$. Now, let's analyze the following equalities step by step,

$$\begin{aligned}
(a_1, \cdots, a_n)e_1 \cdots e_n g_1 \cdots g_{n-1} &= (a_1 e_1 \cdots e_n, \cdots, a_n e_n e_1 \cdots e_{n-1})e_1 \cdots e_n g_1 \cdots g_{n-1} \\
&= (d_1 u_1 e_1 \cdots e_n, \cdots, d_n u_n e_n e_1 \cdots e_{n-1})e_1 \cdots e_n g_1 \cdots g_{n-1} \\
&= (d_1 u_1, \cdots, d_n u_n)e_1 \cdots e_n g_1 \cdots g_{n-1} \\
&= (d_1 u_1 g_1 \cdots g_{n-1}, \cdots, d_n u_n g_1 \cdots g_{n-1})e_1 \cdots e_n g_1 \cdots g_{n-1} \\
&= (d_1 z_1 p g_{n-1}, \cdots, d_n z_n p g_{n-1})e_1 \cdots e_n g_1 \cdots g_{n-1} \ (z_i \in S) \\
&= (d_1 z_1, \cdots, d_n z_n)p e_1 \cdots e_n g_1 \cdots g_{n-1}
\end{aligned}$$

so $(b_1, \cdots, b_n)f_1 \cdots f_n h_1 \cdots h_{n-1} = (d_1 z_1, \cdots, d_n z_n)q f_1 \cdots f_n h_1 \cdots h_{n-1}$. These show that $\Pi_{i=1}^n A_i$ satisfies Condition (P_E). □

Similarly, we obtain the following Theorem 2.11. An S-poset A_S is said to satisfy Condition (PWP_E) if, for all $a, a' \in A_S$, $s \in S$, $as \leqslant a's$ implies the existence of $a'' \in A_S$, $u, v \in S$, $e, f \in E(S)$ such that $ae = a''ue$, $a'f = a''vf$, $es = s = fs$ and $us \leqslant vs$.

Theorem 2.11 *Let S be a commutative pomonoid. The following conditions are equivalent:*

(1) If $A_1, \cdots, A_n$ satisfy Condition (PWP_E) where $n \in \mathbb{N}$, then $\Pi_{i=1}^n A_i$ satisfies Condition (PWP_E);

(2) If A_S and B_S satisfy Condition (PWP_E), then $(A_S \times B_S)$ satisfies Condition (PWP_E);

(3) The diagonal S-poset $D(S)$ satisfies Condition (PWP_E);

(4) For any $s \in S$, the set $L(s,s)$ is either empty or else a weakly locally cyclic idempotent left S-poset.

An S-poset A_S satisfies Condition (P') when, for all $a, a' \in A_S$ and $s, s', z \in S$, $as \leqslant a's'$ and $sz = s'z$ imply there exist $a'' \in A_S$, $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us \leqslant vs'$.

Theorem 2.12 *Let S be a pomonoid and ρ be a right order congruence on S. Then S/ρ satisfies Condition (P') if and only if for any $x, y, s, t, t' \in S$, $xs\rho ys'$ and $sz \leqslant s'z$, imply there exist $u, v \in S$ such that $us = vs'$, $x\rho u$ and $y\rho v$.*

Proof **Necessity.** Assume that S/ρ satisfies Condition (P'). Given any $x, y, s, s', z \in S$ with $xs\rho ys'$ and $sz \leqslant s'z$, we have $[x]_\rho s \leqslant [y]_\rho s' \in S/\rho$. According to (P'), there exist $u', v' \in S$ such that $u's \leqslant v's'$, $[x]_\rho = [z]_\rho u$ and $[y]_\rho = [z]_\rho v$. If $u = zu'$, $v = zv'$, from these, we can claim that $x\rho u$, $y\rho v$ and $us = vs'$.

Sufficiency. Suppose $[x]_\rho s \leqslant [y]_\rho s' \in S/\rho$. For any $x, y, s, t, t' \in S$, $sz \leqslant s'z$. Then $xs\rho ys'$, so by the given assumption, there exist $u, v \in S$ such that $x\rho u$, $y\rho v$ and $us = vs'$. This means, $[x]_\rho \leqslant [1]_\rho u$, $[y]_\rho \leqslant [1]_\rho v$. Therefore, we obtain S/ρ satisfies Condition (P'). □

Let S be a pomonoid and $P \subseteq S$ be sub-pomonoid of S. P is said to be *quasi weakly right reversible*, if for any $s, s' \in S$, $z \in S$, $sz \leqslant s'z$ implies there exist $u, v \in P$ such that $us \leqslant vs'$.

Theorem 2.13 *Let S be a pomonoid. The following conditions are equivalent:*

(1) If the direct product $\Pi_{i \in I} A_i$ satisfies Condition (P'), then each A_i satisfies Condition (P');

(2) Θ_S satisfies Condition (P');

(3) S is a quasi weakly right reversible pomonoid.

Proof (1) ⇒ (2) Since $S \times \Theta_S \cong S$, this implication is straightforward.

(2) ⇒ (3) Given that $S/S_S \cong \Theta_S$, according to Theorem 2.12, $sz \leqslant s'z$ implies for any $s, t, z \in S$, there exist $u, v \in S$ such that $us \leqslant vt$. This means, S is quasi weakly right *reversible* pomonoid.

(3) ⇒ (1) The proof is analogous to that of Theorem 2.7. □

Theorem 2.14 *Let S be a pomonoid. The following conditions are equivalent:*

(1) If $A_1, \cdots, A_n$ satisfy Condition (P') where $n \in \mathbb{N}$, then $\Pi_{i=1}^n A_i$ satisfies Condition (P');

(2) If A_S and B_S satisfy Condition (P'), then $(A_S \times B_S)$ satisfies Condition (P');

(3) The diagonal S-poset $D(S)$ satisfies Condition (P');

(4) For any $a, b \in S$, the set $L(a, b)$ is either empty or else if there exists $z \in S$ such that $(a, b) \in \rho_z$, then $L(a, b)$ is weakly locally cyclic left S-poset.

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (4) Let $L(a, b) \neq \varnothing$ for any $a, b \in S$. Suppose that there exists $z \in S$ such that $(a, b) \in \rho_z$, $(x, y), (x', y') \in L(a, b)$. Then $(x, x')a \leqslant (y, y')b$ and $az \leqslant bz$. By assumption for some $z, z', u, v \in S$ such that $(x, x') = (z, z')u$, $(y, y') = (z, z')v$ and $ua \leqslant vb$. So $(u, v) \in L(a, b)$, $(x, y) = z(u, v)$, $(x', y') = z'(u, v)$. Thus, $L(a, b)$ is a weakly locally cyclic left S-poset.

(4) $\Rightarrow$ (1) Let $(a_1, \cdots, a_n), (a'_1, \cdots, a'_n) \in \Pi_{i=1}^n A_i$ and $s, t, z \in S$. Assume that $(a_1, \cdots, a_n)s \leqslant (a'_1, \cdots, a'_n)t$ and $sz \leqslant tz$. For each i where $1 \leqslant i \leqslant n$, it follows that $a_i s \leqslant a'_i t$ and $sz \leqslant tz$, According to the relevant properties, there exist $b_i \in A_i$ and $u_i, v_i \in S$ such that $a_i = b_i u_i$, $a'_i = b_i v_i$ and $u_i s \leqslant v_i t$. Thus, $(u_i, v_i) \in L(s, t)$, $(s, t) \in \rho_z$. Based on the given assumption, there exists $(u, v) \in L(s, t)$ such that $(u_i, v_i) \in S(u, v)$. And so $us \leqslant vt$ for some $p_i \in S$, $(u_i, v_i) = p_i(u, v)$, then for $1 \leqslant i \leqslant n$, $a_i = b_i u_i = b_i p_i u$, $a'_i = b_i v_i = b_i p_i v$ and $us \leqslant vt$. Consequently, $(a_1, \cdots, a_n) = (b_1 p_1, \cdots, b_n p_n)u$, $(a'_1, \cdots, a'_n) = (b_1 p_1, \cdots, b_n p_n)v$ and $us \leqslant vt$, that is, $\Pi_{i=1}^n A_i$ satisfies Condition (P'). □

Theorem 2.15 *Let S be a pomonoid. The following conditions are equivalent:*

(1) The direct product of every non-empty family of right S-posets satisfying Condition (P') satisfies Condition (P');

(2) $(S^\Gamma)_S$ satisfies Condition (P'), for any nonempty set Γ;

(3) For any $x, y \in S$, the set $L(x, y)$ is either empty or else there exists $z \in S$ such that $(x, y) \in \rho_z$, then $L(x, y)$ is cyclic left S-poset.

Proof (1) $\Rightarrow$ (2) It is clear.

(2) $\Rightarrow$ (3) Consider $x, y \in S$ with $L(x, y) \neq \varnothing$. Assume that there exists an element $z \in S$ such that $(x, y) \in \rho_z$. Let $L(x, y) = \{(s_i, t_i) | i \in I\}$, and define $\overrightarrow{s}, \overrightarrow{t} \in (S^I)_S$ where s_i, t_i are the ith components of $\overrightarrow{s}, \overrightarrow{t}$ respectively. We have $\overrightarrow{s}x \leqslant \overrightarrow{t}y$ in $(S^I)_S$, $xz \leqslant yz$. Given the assumption that $(S^\Gamma)_S$ satisfies Condition (P'), it follows that there exist $u, v \in S$, $\overrightarrow{z} \in (S^\Gamma)_S$ such that $\overrightarrow{s} = \overrightarrow{z}u$, $\overrightarrow{t} = \overrightarrow{z}q$ and $ux \leqslant vy$. Thus, $(u, v) \in L(x, y)$, we have $(s_i, t_i) = z_i(u, v)$, where z_i is the ith component of $\overrightarrow{z}$. Therefore, $L(x, y)$ is a cyclic left S-poset.

(3) $\Rightarrow$ (1) Suppose $A = \Pi_{i \in I} A_i$ is the direct product of a non-empty family of right S-posets, each of which satisfies Condition (P'). Let $(s_i)_I x \leqslant (t_i)_I y$ and $xz \leqslant yz$ for any $(s_i)_I, (t_i)_I \in A$, $x, y, z \in S$, then $s_i a \leqslant t_i b$, $xz \leqslant yz$. Since A_i satisfies Condition (P'), implies there exist $z_i \in A_i$, $u_i, v_i \in S$ such that $s_i = z_i u_i$, $t_i = z_i v_i$ and $u_i x \leqslant v_i y$. Then $L(x, y) \neq \varnothing$, $(x, y) \in \rho_z$, by assumption there exists $(u, v) \in L(x, y)$ such that $(u_i, v_i) \in S(u, v)$, $i \in I$. Furthermore, we can derive $ux \leqslant vy$, $(u_i, v_i) = w_i(u, v)$ for some $w_i \in S$, so $(s_i)_I = (z_i w_i)_I u$, $(t_i)_I = (z_i w_i)_I v$. Therefore, A satisfies Condition (P') as required. □

Here we define that a right S-poset A_S satisfies *Condition* (E') when, for all $a \in A_S$, $s, s', z \in S$, $as \leqslant as'$ and $sz \leqslant s'z$ imply there exist $a' \in A_S$, $u \in S$ such that $a = a'u$ and $us \leqslant us'$.

Let S be a pomonoid and $P \subseteq S$ be sub-pomonoid of S. We say that P isquasi weakly left *collapsible*, if for any $s, s' \in S$, $z \in S$, $sz \leqslant s'z$ implies there exists $u \in P$ such that $us \leqslant us'$.

Theorem 2.16 *Let S be a pomonoid. The following conditions are equivalent:*

(1) If $\Pi_{i\in I}A_i$ satisfies Condition (E'), then A_i satisfies Condition (E');

(2) Θ_S satisfies Condition (E');

(3) S is a quasi weakly left collapsible pomonoid.

Proof It is similar to Theorem 2.7. □

Theorem 2.17 *Let S be a pomonoid. The following conditions are equivalent:*

(1) The direct product of every non-empty family of right S-posets satisfying Condition (E') satisfies Condition (E');

(2) $(S^{\Gamma})_S$ satisfies Condition (E'), for any nonempty set Γ;

(3) For any $x, y \in S$, the set $l(x, y) := \{s \in S : sx \leqslant sy\}$ is either empty or else there exists $z \in S$ such that $(x, y) \in \rho_z$, then $l(x, y)$ is a principally left ideal of S.

Proof It is similar to [13, Theorem 3.23]. □

Theorem 2.18 *Let S be a pomonoid. The following conditions are equivalent:*

(1) If $A_1, \cdots, A_n$ satisfy Condition (E') where $n \in \mathbb{N}$, then $\Pi_{i=1}^{n}A_i$ satisfies Condition (E');

(2) If A_S and B_S satisfy Condition (E'), then $(A_S \times B_S)$ satisfies Condition (E');

(3) The diagonal S-poset $D(S)$ satisfies Condition (E');

(4) For any $x, y \in S$, the set $l(x, y)$ is either empty or else there exists $z \in S$ such that $(x, y) \in \rho_z$, then $l(x, y)$ is weakly locally principally.

Theorem 2.19 *Let $S = T^1$, where T is a left po-group. For each $i \in I$, if A_i satisfies Condition (E), then the direct product $\Pi_{i\in I}A_i$ also satisfies Condition (E).*

Proof Assume that $(a_i)_I s \leqslant (a_i)_I s'$ where $(a_i) \in \Pi_{i\in I}A_i$ and $s, s' \in S$. This implies $a_i s \leqslant a_i s'$ for every $i \in I$. According to the given assumption, for each i, there exist $u_i \in S$, $a'_i \in A_i$ such that $a_i = a'_i u_i$, $u_i s \leqslant u_i s'$. Now, consider two cases. If $u_j = 1$, then $s \leqslant s'$ for some $j \in I$. Otherwise, fix an element $k \in I$. Then for any $j \in I$, $j \neq k$, there exists $x_j \in T$ such that $u_j = x_j u_k$. Let $b_k = a'_k$ and for any $j \neq k$, define $b_j = a'_j x_j$, then $(a_i)_I = (a'_i u_i)_I = (b_i)_I u_k$. □

Theorem 2.20 *Let $S = T^1$, where T is a left po-group. For each $i \in I$, if A_i satisfies Condition (E'), then the direct product $\Pi_{i\in I}A_i$ also satisfies Condition (E').*

Proposition 2.21 *Let S be any commutative pomonoid. If $\Pi_{i\in I}A_i$ satisfies Condition (P) (or Condition (P'), Condition (P_E)), then A_i also satisfy Condition (P) (or Condition (P'), Condition (P_E)).*

If I is any non-empty set and S is any pomonoid, for any element $\vec{a} = (a_i)i \in I \in S^I$, we define

$$L(\vec{a}) = \{\vec{s} \in S^I : s_i a_i \leqslant s_j a_j, \quad \forall i, j \in I\},$$

and

$$l(\vec{a}) = \{s \in S : s a_i \leqslant s a_j, \quad \forall i, j \in I\}.$$

Clearly, if non-empty, $L(\vec{a})$ and $l(\vec{a})$ are a left S-subposet of S^I and a left ideal of S, respectively.

Proposition 2.22 *Let S be a pomonoid such that the diagonal S-poset $D(S)$ is projective. Then, for every non-empty set I and every $\vec{a} \in S^I$, $L(\vec{a})$ is either empty or a weakly locally cyclic S-subposet of S^I, and $l(\vec{a})$ is either empty or else a weakly locally principal left ideal of S.*

Proof First, we focus on the set $L(\vec{a})$. Let $\vec{x}, \vec{y} \in L(\vec{a})$ such that $x_i a_i \leqslant x_k a_k$ and $y_i a_i \leqslant y_k a_k$ for any $i, k \in I$. We start with the following manipulation:

$$(x_i, y_i) \nparallel (x_i, y_i) \cdot 1$$

$$(x_i, y_i) a_i \nparallel (x_k, y_k) \cdot a_k$$

$$(x_k, y_k) \cdot 1 \nparallel (x_k, y_k).$$

Consequently, all (x_k, y_k) belong to a single strongly convex connected component of $D(S)$. Since the diagonal S-poset $D(S)$ is projective, there exists $e^2 = e \in E(S)$ such that in the component forms $(p, q)S$, (p, q) is left e-pocancellative. This means, for any $u, v \in S$ such that $(p, q) = (p, q)e$, from $(p, q)u \leqslant (p, q)v$ implies $eu \leqslant ev$. So $(x_i, y_i) = (p, q)z_i$ for some $z_i \in S$. We need verify $e\vec{z} \in L(\vec{a})$. Furthermore, $\vec{x}, \vec{y} \in Se\vec{z}$. Note that $(p, q)z_i a_i = (x_i, y_i)a_i \leqslant (x_k, y_k)a_k = (p, q)z_k a_k$. We get $ez_i a_i \leqslant ez_k a_k$ for any $i, k \in I$ since left e-pocancellative, $e\vec{z} \in L(\vec{a})$. So $\vec{x} = p\vec{z} = pe\vec{z} \in Se\vec{z}$, $\vec{y} = q\vec{z} = qe\vec{z} \in Se\vec{z}$. Therefore $L(\vec{a})$ is weakly locally cyclic.

Now assume that $x, y \in l(\vec{a})$. Since the diagonal S-poset $D(S)$ is projective, $(x, y) \in (p, q)S$ and (p, q) is left e-pocancellative, then $(x, y) = (p, q)z = (p, q)ez$, and so $(p, q)za_i = (xa_i, ya_i) \leqslant (xa_k, ya_k) = (p, q)za_k$. Then $x, y \in Sez$, $l(\vec{a})$ is a weakly locally principal left ideal of S. □

3. Condition (P_I), $GP-(P)$, strongly (P)-cyclic and I-inverse of diagonal S-posets $D(S)$

In this section, we initially address the query regarding the circumstances under which the diagonal S-poset $D(S)$ meets Condition (P_I) and $GP-(P)$. Subsequently, we demonstrate the equivalent conditions for determining when the (finite) direct products of strongly (P)-cyclic S-posets are strongly (P)-cyclic. Lastly, we characterize the pomonoid S for which $D(S)$ is I-inverse.

Let?I?denote an ideal of the pomonoid?S. A right S-poset A_S satisfies Condition (P_I) when, if $as \leqslant a's'$ for any $a, a' \in A_S$, $s, s' \in S$, then there exist $a'' \in A_S$, $u, v \in S$ such that $us \leqslant vs'$, $a = a''u$, $a' = a''v$. Now, consider that a subset $R \subseteq S$ is right $I-reversible$, if for any $p, q \in S$, there exist $u, v \in R \cap I$ such that $up \leqslant vq$. In particular, right $I - reversible$ is right $reversible$ as $I = S$.

Definition 3.1 *Let $I \subseteq S$ is an ideal. A right S-poset A_S is said to be I-*weakly locally cyclic *if, for any $x, y \in A_S$, there exist $z \in A \cap I$, $s, t \in S$ such that $x = sz$, $y = tz$. In particular, I-weakly locally cyclic is weakly locally cyclic as $I = S$.*

Definition 3.2 *Let $I \subseteq S$ is an ideal. The set $L(a, b)$ is said to be I-*cyclic *if, for any $x \in L(a, b)$, there exist $z \in L(a, b) \cap (I \times I)$, $s \in S$ such that $x = sz$. In particular, I-cyclic is cyclic as $I = S$.*

Theorem 3.3 *Let S be a pomonoid and S satisfies Condition (P_I). The following conditions are equivalent:*

(1) If $A_1, \cdots, A_n$ satisfy Condition (P_I) where $n \in \mathbb{N}$, then $\Pi_{i=1}^n A_i$ satisfies Condition (P_I);

(2) If A_S and B_S satisfy Condition (P_I), then $(A_S \times B_S)$ satisfies Condition (P_I);

(3) The diagonal S-poset $D(S)$ satisfies Condition (P_I);

(4) For any $a, b \in S$, the set $l(a, b)$ is either empty or else I-weakly locally cyclic.

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (4) Assume that the diagonal S-poset $D(S)$ satisfies Condition (P_I). Let $(x, y), (x', y') \in L(a, b)$ with $a, b \in S$, then $xa \leqslant yb$ and $x'a \leqslant y'b$, and so we have $(x, x')a \leqslant (x, x')b \in D(S)$. Based on the given assumption, there must exist $(z, z') \in D(S)$ and $p, q \in I$ such that $(x, x') = (z, z')p$, $(y, y') = (z, z')q$ and $pa \leqslant qb$. From the above equalities, we can further derive that $(x, y) = (zp, zq) = z(p, q)$, $(x', y') = (z'p, z'q) = z'(p, q)$ with $(p, q) \in L(a, b) \cap (I \times I)$. This clearly shows that $L(a, b)$ is I-weakly locally cyclic.

(4) $\Rightarrow$ (1) Suppose that $A_1, \cdots, A_n$ satisfy Condition (P_I). Let $(a_1, \cdots, a_n)u \leqslant (a'_1, \cdots, a'_n)v$ in $A = \Pi_{i=1}^n A_i$ for any $a_i, a'_i \in A_i$, $i \in \mathbb{N}$, $u, v \in S$. From $a_i u \leqslant a'_i v$ and Condition (P_I), there exist $a''_i \in A_i$, $p_i, q_i \in I$ such that $a_i = a''_i p_i$, $a'_i = a''_i q_i$ and $p_i u \leqslant q_i v$. Then $(p_i, q_i) \in L(u, v)$, so by assumption, there exist $(p, q) \in L(u, v) \cap (I \times I)$, $w_i \in S$ such that $(p_i, q_i) = w_i(p, q)$, $i \in \mathbb{N}$. Thus, we have

$$(a_1, \cdots, a_n) = (a''_1 p_1, \cdots, a''_n p_n) = (a''_1 w_1 p, \cdots, a''_n w_n p) = (a''_1 w_1, \cdots, a''_n w_n)p,$$

$$(a'_1, \cdots, a'_n) = (a''_1 q_1, \cdots, a''_n q_n) = (a''_1 w_1 q, \cdots, a''_n w_n q) = (a''_1 w_1, \cdots, a''_n w_n)q,$$

and $up \leqslant vq \in A = \Pi_{i=1}^n A_i$. Thus, $A = \Pi_{i=1}^n A_i$ satisfies Condition (P_I). □

Theorem 3.4 *Let S be a pomonoid and S satisfies Condition (P_I). The following conditions are equivalent:*

(1) The direct product of every non-empty family of right S-posets satisfying Condition (P_I) satisfies Condition (P_I);

(2) $(S^\Gamma)_S$ satisfies Condition (P_I), for any nonempty set Γ;

(3) For any $a, b \in S$, the set $L(a,b)$ is either empty or else I-cyclic.

Proof (1) $\Rightarrow$ (2) It is clear.

(2) $\Rightarrow$ (3) Suppose that $a, b \in S$, $L(a,b) \neq \varnothing$ and $L(a,b) := \{(u_\gamma, v_\gamma) | \gamma \in \Gamma\}$. Let $\overrightarrow{u}, \overrightarrow{v}$ be the elements of S^Γ whose γth components are u_γ, v_γ respectively. Then $\overrightarrow{u}a \leqslant \overrightarrow{v}b$ in S^Γ. Since $(S^\Gamma)_S$ satisfies Condition (P_I), so there exist $\overrightarrow{z} \in S^\Gamma$, $p, q \in I$ such that $\overrightarrow{u} = \overrightarrow{z}p$, $\overrightarrow{v} = \overrightarrow{z}q$ and $pa \leqslant qb$. Thus, $(p,q) \in L(a,b) \cap (I \times I)$ so that $(p,q) = (u_j, v_j)$ for some $j \in \Gamma$. If $\gamma \in \Gamma$, then $(u_\gamma, v_\gamma) = z_\gamma(p,q) = z_\gamma(u_j, v_j)$, where z_γ is the γth component of $\overrightarrow{z}$, Thus, $L(a,b)$ is I-cyclic.

(3) $\Rightarrow$ (1) Let $A = \Pi_{i\in I} A_i$ is the direct product of every non-empty family of right S-posets satisfying Condition (P_I). Suppose that $\overrightarrow{x}a \leqslant \overrightarrow{y}b$ for any $a, b \in S$, $\overrightarrow{x} = (x_i)$, $\overrightarrow{y} = (y_i) \in A$. For any $i \in I$, $x_i a \leqslant y_i b$. Since A_i satisfies Condition (P_I), so there exist $u_i, v_i \in I$, $z_i \in A_i$ such that $u_i a \leqslant v_i b$, $x_i = z_i u_i$, $y_i = z_i v_i$. And so $(u_i, v_i) \in L(a,b) \neq \varnothing$. By assumption, $L(a,b)$ is I-cyclic, there exist $(s,t) \in L(a,b) \cap (I \times I)$, $w_i \in S$ such that $(u_i, v_i) = w_i(s,t)$. Thus, we have

$$(x_i)_I = (z_i u_i)_I = (z_i w_i s)_I = (z_i w_i)_I s,$$

$$(y_i)_I = (z_i v_i)_I = (z_i w_i)_I t,$$

and $sa \leqslant tb \in A$. Therefore, $A = \Pi_{i\in I} A_i$ satisfies Condition (P_I). □

Definition 3.5 *A right S-poset A_S satisfies* Condition $GP-(P)$, *if $as \leqslant a's$ for any $a, a' \in A_S$, $s \in S$, then there exist $n \in \mathbb{N}$, $a'' \in A_S$, $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us^n \leqslant vs^n$.*

We define the set $L(s^n, s^n) := \{(u,v) \in D(S) | us^n \leqslant vs^n\}$ for any $n \in \mathbb{N}$. Obviously, $L(s^n, s^n)$ is a left S-poset. The following Theorem gives the characterization while the diagonal S-poset $D(S)$ satisfies Condition $GP-(P)$.

Theorem 3.6 *Let S be a pomonoid. The following conditions are equivalent:*

(1) The diagonal S-poset $D(S)$ satisfies Condition $GP-(P)$;

(2) For any $s \in S$, the set $L(s,s)$ is either empty or else satisfies the Condition:

for any $(u,v), (u',v') \in L(s,s)$, there exist $n \in \mathbb{N}$, $(p,q) \in L(s^n, s^n)$ such that $(u,v), (u',v') \in S(p,q)$.

Proof (1) $\Rightarrow$ (2) Suppose that the diagonal S-poset $D(S)$ satisfies Condition $GP-(P)$. Let for any $(u,v), (u',v') \in S(p,q)$, $s \in S$. Then $us \leqslant vs$, $u's \leqslant v's$. It follows that $(u,u')s \leqslant (v,v')s$. By assumption, there exist $n \in \mathbb{N}$, $(w,w') \in D(S)$, $p, q \in S$ such that $(u,u') = (w,w')p$, $(v,v') = (w,w')q$ and $ps^n \leqslant qs^n$, so we obtain $(u,v), (u',v') \in S(p,q)$ and $(p,q) \in L(s^n, s^n)$ as required.

(2) $\Rightarrow$ (1) Suppose that $(a,b), (a',b') \in D(S)$, $s \in S$ such that $(a,b)s \leqslant (a',b')s$. Then we have $as \leqslant a's$, $bs \leqslant b's$. Thus, $(a,a'), (b,b') \in L(s,s) \neq \varnothing$. By assumption, there exist $n \in \mathbb{N}$, $(p,q) \in L(s^n, s^n)$ such that $(a,a'), (b,b') \in S(p,q)$. Then $(a,a') = w(p,q)$,

$(b,b')=w'(p,q)$ and $ps^n \leqslant qs^n$ for some $w,w' \in S$. Therefore, we get $(a,b)=(w,w')p$, $(a',b')=(w,w')q$. So the diagonal S-poset $D(S)$ satisfies Condition $GP-(P)$. □

Theorem 3.7 *Let S be a pomonoid. The following conditions are equivalent:*

(1) $(S^\Gamma)_S$ satisfies Condition $GP-(P)$ for any nonempty set Γ;

(2) For any $s \in S$, the set $L(s,s)$ is either empty or else there exist $n \in \mathbb{N}$, $(p,q) \in L(s^n,s^n)$ such that $L(s,s) \subseteq S(p,q)$.

Proof (1) ⇒ (2) Let $s \in S$, $L(s,s) \neq \varnothing$ and $L(s,s)=\{(u_\gamma,v_\gamma)|\gamma \in \Gamma\}$. Suppose that $\overrightarrow{u}, \overrightarrow{v}$ are elements of S^Γ whose γth components are u_γ, v_γ, respectively. Then $\overrightarrow{u}s \leqslant \overrightarrow{v}s$ in S^Γ. By assumption, $(S^\Gamma)_S$ satisfies Condition $GP-(P)$, there exist $n \in \mathbb{N}$, $\overrightarrow{z} \in S^\Gamma$, $p,q \in S$ such that $\overrightarrow{u}=\overrightarrow{z}p$, $\overrightarrow{v}=\overrightarrow{z}q$ and $ps^n \leqslant qs^n$. Thus, $(p,q) \in L(s^n,s^n)$. Suppose $\overrightarrow{z}=(z_\gamma)_{\gamma\in\Gamma}$, then we have $u_\gamma=z_\gamma p$, $v_\gamma=z_\gamma q$ for any $\gamma \in \Gamma$. Therefore $(u_\gamma,v_\gamma)=z_\gamma(p,q)$ for any $\gamma \in \Gamma$, it is follows that $L(s,s) \subseteq S(p,q)$.

(2) ⇒ (1) Suppose that $\overrightarrow{u}s \leqslant \overrightarrow{v}s$ for $\overrightarrow{u},\overrightarrow{v} \in S^\Gamma$, $s \in S$. Let $\overrightarrow{u}=(u_\gamma)_{\gamma\in\Gamma}$, $\overrightarrow{v}=(v_\gamma)_{\gamma\in\Gamma}$, then $u_\gamma s \leqslant v_\gamma s$ for any $\gamma \in \Gamma$, so $(u_\gamma,v_\gamma) \in L(s,s) \neq \varnothing$. By assumption, there exist $n \in \mathbb{N}$, $(p,q) \in L(s^n,s^n)$ such that $L(s,s) \subseteq S(p,q)$. Thus, for any $\gamma \in \Gamma$, there exists $w_\gamma \in S$ such that $(u_\gamma,v_\gamma)=w_\gamma(p,q)$. Put $\overrightarrow{w}=(w_\gamma)_{\gamma\in\Gamma}$. Then we have $\overrightarrow{u}=\overrightarrow{w}p$, $\overrightarrow{v}=\overrightarrow{w}q$ and $ps^n \leqslant qs^n$, so $(S^\Gamma)_S$ satisfies Condition $GP-(P)$. □

Recall that a pomonoid S is *a right PCP pomonoid*, if all right principally ideal of S satisfy Condition (P). A right S-poset A_S is *strongly (P)-cyclic*, if for any $a \in A_S$, there exists $z \in S$ such that $\overrightarrow{ker}\lambda_a=\overrightarrow{ker}\lambda_z$ and zS satisfies Condition (P).

Theorem 3.8 *The diagonal S-poset $D(S)$ is strongly (P)-cyclic if and only if the following conditions hold:*

(1) S is a right PCP pomonoid;

(2) $L=\{\overrightarrow{ker}\lambda_z|zS$ satisfies Condition $(P)\} \cup \{S\times S\}$ is a sub-pomonoid of $T=(ConS_S,\cap)$.

Proof **Necessity.** Take $s \in S$. Then by assumption there exists $z \in S$ such that $\overrightarrow{ker}\lambda_s=\overrightarrow{ker}\lambda_{(s,s)}\overrightarrow{ker}\lambda_z$ and zS satisfies Condition (P). Thus, S is a right PCP pomonoid. On the other hand, Suppose that z_1S and z_2S satisfy Condition (P) for any $z_1,z_2 \in S$. Then there exists $z \in S$ such that

$$\overrightarrow{ker}\lambda_{z_1} \cap \overrightarrow{ker}\lambda_{z_2}=\overrightarrow{ker}\lambda_{(z_1,z_2)}=\overrightarrow{ker}\lambda_z$$

and zS satisfies Condition (P).

Sufficiency. Let $(s,t) \in D(S)$. Since S is a right PCP pomonoid, then sS and tS satisfy Condition (P). Then by assumption there exists $z \in S$ such that $\overrightarrow{ker}\lambda_s \cap \overrightarrow{ker}\lambda_t=\overrightarrow{ker}\lambda_z$ and zS satisfies Condition (P). From $\overrightarrow{ker}\lambda_{(s,t)}=\overrightarrow{ker}\lambda_s \cap \overrightarrow{ker}\lambda_t=\overrightarrow{ker}\lambda_z$, then the diagonal S-poset $D(S)$ is strongly (P)-cyclic. □

Theorem 3.9 *Let S be a right PCP pomonoid. The following conditions are equivalent:*

(1) The finite product of strongly (P)-cyclic S-posets is strongly (P)-cyclic;

(2) S^n is strongly (P)-cyclic, for any $n \in \mathbb{N}$;

(3) The diagonal S-poset $D(S)$ is strongly (P)-cyclic.

Proof (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (1) Suppose that A_S and B_S are strongly (P)-cyclic. Let $(a,b) \in A \times B$. Then there exist $z_1, z_2 \in S$ such that $\overrightarrow{ker}\lambda_a = \overrightarrow{ker}\lambda_{z_1}$, $\overrightarrow{ker}\lambda_b = \overrightarrow{ker}\lambda_{z_2}$, z_1S and z_2S satisfy Condition (P). By Theorem 3.8, zS satisfies Condition (P) for some $z \in S$ and

$$\overrightarrow{ker}\lambda_{(a,b)} = \overrightarrow{ker}\lambda_a \cap \overrightarrow{ker}\lambda_b = \overrightarrow{ker}\lambda_{z_1} \cap \overrightarrow{ker}\lambda_{z_2} = \overrightarrow{ker}\lambda_z.$$

Thus, $(A_S \times B_S)$ is strongly (P)-cyclic. Finally, we use induction to obtain conclusion. □

Theorem 3.10 *Let S be a right PCP pomonoid. The following conditions are equivalent:*

(1) The direct product of non-empty family of strongly (P)-cyclic S-posets is strongly (P)-cyclic;

(2) $(S^\Gamma)_S$ is strongly (P)-cyclic, for any non-empty Γ;

(3) For any family of $\{z_i | z_iS$ satisfies Condition $(P)\}$, there exists $z \in S$ such that zS satisfies Condition (P) and $\cap_{i\in I}\overrightarrow{ker}\lambda_{z_i} = \overrightarrow{ker}\lambda_z$.

Since [18] introduced the definition of I-regular S-posets. Let A be an S-poset. an element $a \in A$ is called *I-regular* if there exists an S-morphism $f : aS \to S$ such that $af(a) = a$. An S-poset A is called *I-regular* if all elements of A are I-regular.

Definition 3.11 *Let A be an S-poset. an element $a \in A$ is called* I-inverse *if there exists a unique S-morphism $f : aS \to S$ such that $af(a) = a$. An S-poset A is called I-inverse if all elements of A are I-inverse.*

Proposition 3.12 *Let A be an S-poset and $a \in A$. Then the following statements are equivalent:*

(1) a is I-inverse;

(2) There exists an unique element $e \in E(S)$ such that $a = ae$ and $ap \leqslant aq$ implies $ep \leqslant eq$ for $p, q \in S$.

Proof (1) $\Rightarrow$ (2) Let a is an I-inverse element of A, then a is an I-regular element. It satisfies condition (2) of [18, Proposition 4.2]. Suppose that there exists an idempotent $h \neq e$ such that $a = ah$, and for any $p, q \in S$, from $ap \leqslant aq$ implies $hp \leqslant hq$. We define a map $g : aS \to S$ by $g(as) = hs$. Clearly g is well done and a right S-morphism. But then $g(a) = h$, $ag(a) = a$. This contradicts the I-inverse element.

(2) $\Rightarrow$ (1) Define a map $f : aS \to S$ by $f(as) = es$, $s \in S$. It is easy to see f is a morphism and $af(a) = a$. Suppose that another morphism $g : aS \to S$ by $ag(a) = a$. Put $g(a) = h$, by assumption $h \neq e$. Then $a = ah$ by [18, Proposition 4.2], for $p, q \in S$, $ap \leqslant aq$ implies $hp \leqslant hq$, it contradicts with (2). □

Definition 3.13 *A pomonoid S is said to be* right UPP pomonoid *if for any $s \in S$, there exists a unique idempotent $e \in E(S)$ such that $s = se$, and from $sx \leqslant sy$ implies $ex \leqslant ey$ for $x, y \in S$.*

It is known that A_S is I-inverse if and only if for every $a \in A$ there exists a unique idempotent $e \in S$ such that $\overrightarrow{ker}\lambda_a = \overrightarrow{ker}\lambda_e$. A pomonoid S is a right UPP pomonoid if and only if for each $s \in S$, $\overrightarrow{ker}\lambda_s = \overrightarrow{ker}\lambda_e$ for a unique idempotent $e \in S$.

The next theorem gives a characterization of pomonoids over which $D(S)$ is I-inverse.

Theorem 3.14 *Let S be a pomonoid. The diagonal S-poset $D(S)$ is I-inverse if and only if:*

(1) S is right UPP;

(2) The set $R = \{\overrightarrow{ker}\lambda_e | e \in E(S)\} \cup \{S \times S\}$ is a subpomonoid of $T = (ConS_S, \cap)$.

Proof **Necessity.** Take $s \in S$. Since the diagonal S-poset $D(S)$ is I-inverse, $\overrightarrow{ker}\lambda_s = \overrightarrow{ker}\lambda_{(s,s)} = \overrightarrow{ker}\lambda_e$ for a unique idempotent $e \in S$. Thus S is right UPP pomonoid. On the other hand, by assumption for each pair of idempotents $e, f \in S$ such that $\overrightarrow{ker}\lambda_e \cap \overrightarrow{ker}\lambda_f = \overrightarrow{ker}\lambda_{(e,f)}$. There exists a unique idempotent $h \in S$ such that $\overrightarrow{ker}\lambda_{(e,f)} = \overrightarrow{ker}\lambda_h$ from I-inverse which completes the proof of necessity.

Sufficiency. Let $(s,t) \in D(S)$ for $s, t \in S$. So there exists a unique idempotent $e \in E(S)$ such that $\overrightarrow{ker}\lambda_s = \overrightarrow{ker}\lambda_e$ and there exists a unique idempotent $f \in E(S)$ such that $\overrightarrow{ker}\lambda_t = \overrightarrow{ker}\lambda_f$. Since R is a subpomonoid of T, there exits an idempotent $h \in S$ such that $\overrightarrow{ker}\lambda_e \cap \overrightarrow{ker}\lambda_f = \overrightarrow{ker}\lambda_{(e,f)} = \overrightarrow{ker}\lambda_h$. By assumption, S is right UPP, $\overrightarrow{ker}\lambda_h = \overrightarrow{ker}\lambda_g$ for a unique idempotent $g \in E(S)$. Thus

$$\overrightarrow{ker}\lambda_{(s,t)} = \overrightarrow{ker}\lambda_s \cap \overrightarrow{ker}\lambda_t = \overrightarrow{ker}\lambda_e \cap \overrightarrow{ker}\lambda_f = \overrightarrow{ker}\lambda_{(e,f)} = \overrightarrow{ker}\lambda_h = \overrightarrow{ker}\lambda_g.$$

Therefore, the diagonal S-poset $D(S)$ is I-inverse. □

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